Pergamon

www.elsevier.com/locate/jappmathmech

PII: S0021-8928(02)00073-4

J. Appl. Maths Mechs, Vol. 66, No. 4, pp. 547–558, 2002

0021-8928/02/$—see front matter

OUTER POLYHEDRAL ESTIMATES FOR ATTAINABILITY SETS OF SYSTEMS WITH BILINEAR UNCERTAINTY†

E. K. KOSTOUSOVA

Ekaterinburg

e-mail: kek@imm.uran.ru

(*Received* 4 *June* 2001)

For the problem of finding attainability sets of systems with bilinear uncertainty, the possibilities of outer approximations (estimates) by parallelepipeds are investigated. Evolution equations describing the dynamics of outer estimates are derived. Results of a numerical simulation are presented. © 2002 Elsevier Science Ltd. All rights reserved.

The problem of constructing trajectory tubes (many-valued functions describing the dynamics of attainability sets, solvability sets, and information domains) may be considered to be one of the fundamental problems of mathematical control theory (see, e.g. [1–4]). The exact construction of these sets may prove difficult. The attainability sets of systems with bilinear uncertainty considered in this paper (linear systems with indeterminate matrices) [5–8] need not be be convex. Therefore, along with other approximation methods, it seems important to develop methods to construct simple but efficient estimates for attainability sets. Since the initial many-valued functions possess the semigroup property, one naturally requires the estimates to possess an analogous property. One of the most rapidly developing methods in this area is that of ellipsoidal approximation (see, e.g. [3, 4, 7, 9–11]). Parallelepipedal estimates are also being constructed (see, e.g. [1, 9, 12, 13] and the bibliographies cited there). Coordinatewise estimates may be obtained by using interval calculus [14, 15]. However, these may turn out to be too coarse, because of the well-known "wrapping effect" [15, p. 177].

Below we present ordinary differential equations which, for given dynamics of the orientation matrices, describe the dynamics of centres and "semi-axis values" for two types of parallelepipedal estimates of attainable sets. Simpler estimates of the first type are constructed on the basis of approximations analogous to those proposed in [7, 11]; those of the second type are based on more accurate approximations. Model examples are used to compare these estimates with ellipsoidal estimates [7, 10]. It is observed that they may be less laborious and more accurate (but not always). A case is singled out in which more accurate estimates for attainability sets may be obtained by combining several parallelepipedal estimates.

1. FORMULATION OF THE PROBLEM

Suppose the state $x \in \mathbb{R}^n$ of an object is described by the system

$$\dot{x} = A(t)x + B(t)w(t), \quad t \in T = [0, \theta] \tag{1.1}$$

where $\mathbb{R}^n$ Euclidean n-space; the initial state $x(0) = x_0$ and action $w(\cdot)$ (a Lebesgue-measurable n-dimensional function of time t) are unknown in advance and subject to restrictions

$$x(0) \in \mathbf{X}_0, \quad w(t) \in \mathbf{R}(t) \text{ for almost all } t \in T \tag{1.2}$$

$\mathbf{X}_0$ and $\mathbf{R}(t)$ are given convex compact sets and the multivalued mapping $\mathbf{R}(t)$ is continuous; then $n \times n$ matrix-valued functions A and B are also not known exactly but subject to the restrictions

$$A(t) \in \mathbf{A}(t), \quad \mathbf{A}(t) = \{A : \underline{A}(t) \leqslant A \leqslant \overline{A}(t)\}, \quad t \in T \tag{1.3}$$

$$B(t) \in \mathbf{B}(t), \quad \mathbf{B}(t) = \{B : \underline{B}(t) \leqslant B \leqslant \overline{B}(t)\}, \quad t \in T \tag{1.4}$$

†*Prikl. Mat. Mekh.* Vol. 66, No. 4, pp. 559–571, 2002.

Relations (1.3) may be written differently as

$$A(t)\in\mathbf{A}(t)=\{A:A=\tilde{A}(t)+\Delta A(t),\quad \Delta A(t)\in\hat{\mathbf{A}}(t)\}$$
$$\hat{\mathbf{A}}(t)=\{A:\operatorname{Abs}A\leqslant\hat{A}(t)\},\quad \tilde{A}=(\underline{A}+\overline{A})/2,\quad \hat{A}=(\overline{A}-\underline{A})/2 \tag{1.5}$$

Matrix inequalities, here and below, should be understood in componentwise fashion. Abs$A$ denotes the matrix of absolute values of the elements of the matrix $A = \{a_i^j\}$: Abs$A = \{|a_i^j|\}$ (the superscript indicates the columns, the subscript the rows). The notation $\tilde{B},\hat{B}$ and $\hat{\mathbf{B}}$ is defined similarly. It is assumed that the known functions $\underline{A} = \{\underline{a}_i^j\}$, $\overline{A} = \{\overline{a}_i^j\}$, $\underline{B} = \{\underline{b}_i^j\}$, $\overline{B} = \{\overline{b}_i^j\}$ are continuous in t.

The *attainability set* $\mathbf{X}(t) = \mathbf{X}(t, 0, \mathbf{X}_0)$ of system (1.1)–(1.4) for $t \geqslant 0$ is the set of points $x \in \mathbb{R}^n$ for each of which x_0, $w(\cdot)$, $A(\cdot)$ and $B(\cdot)$ exist satisfying conditions (1.2)–(1.4) and generating a solution $x(\cdot)$ of system (1.1) such that $x(t) = x$.

Attainability sets are known to possess the *semigroup property*

$$\mathbf{X}(t,0,\mathbf{X}_0)=\mathbf{X}(t,\tau,\mathbf{X}(\tau,0,\mathbf{X}_0)),\quad \forall\tau,t:\ 0\leqslant\tau\leqslant t\leqslant\theta \tag{1.6}$$

We shall assume that $\mathbf{X}_0$ and $\mathbf{R}(t)$ are parallelepipeds and seek outer parallelepipedal estimates $\mathbf{P}(t)$ for $\mathbf{X}(t)$

$$\mathbf{X}(t)\subseteq\operatorname{co}\mathbf{X}(t)\subseteq\mathbf{P}(t),\quad \forall t\in T \tag{1.7}$$

A *parallelepiped* $\mathbf{P}(p, P, \pi)$ in $\mathbb{R}^n$ is a set

$$\mathbf{P}=\mathbf{P}(p,P,\pi)=\{x:x=p+\sum_{i=1}^{n}p^i\pi_i\xi_i,\quad |\xi_i|\leqslant 1,\quad i=1,\dots,n\}$$
$$p\in\mathbb{R}^n;\quad P=\{p_j^i\}=\{p^1\dots p^n\}\in M_*^{n\times n};\quad \pi\in\mathbb{R}^n,\quad \pi\geqslant 0$$

where $M_*^{n\times n} = \{P\colon \det P \neq 0,\ \|p^i\| = 1, i = 1,\dots,n\}$ is the set of all non-singular $n \times n$ matrices P with columns p^i of unit length ($\|x\| = (x,x)^{1/2}$ is the Euclidean norm). One can say that p defines the centre of the parallelepiped, p^i are "directions," and π_i are the lengths of its "semiaxes". Note that the condition $\|p^i\| = 1$ is not essential and may be omitted.

We are thus assuming that

$$\mathbf{X}_0=\mathbf{P}(p_0,P_0,\pi_0),\quad \mathbf{R}(t)=\mathbf{P}(r(t),R(t),\rho(t)) \tag{1.8}$$

where r, R and ρ are continuous vector- and matrix-valued functions. We shall seek outer estimates $\mathbf{P}(t) = \mathbf{P}(p(t), P(t), \pi(t))$ for $\mathbf{X}(t)$ possessing generalized semigroup and evolution properties, which are analogues of property (1.6). We recall that the *evolution property* [4] of estimates is formulated in terms of attainability sets

$$\mathbf{X}(t,\tau,\mathbf{P}(\tau))\subseteq\mathbf{P}(t),\quad \forall\tau,t:\ 0\leqslant\tau\leqslant\theta;\quad \mathbf{X}_0\subseteq\mathbf{P}(0) \tag{1.9}$$

and guarantees that (1.7) is satisfied. As we shall see below, $\mathbf{P}(t)$ may be found from the evolution equations with initial conditions $\mathbf{P}(0)$, and by analogy with attainability sets one can introduce the notation $\mathbf{P}(t) = \mathbf{P}\{(t, 0, \mathrm{P}(0)\}$. We say [3] that estimates $\mathbf{P}(t)$ possess the *"upper" semigroup property* if

$$\mathbf{P}\{t,0,\mathbf{P}(0)\}=\mathbf{P}\{t,\tau,\mathbf{P}\{\tau,0,\mathbf{P}(0)\}\},\quad 0\leqslant\tau\leqslant t\leqslant\theta; \tag{1.10}$$

In what follows we shall use the following notation: co$\mathbf{X}$ is the convex hull of $\mathbf{X} \subseteq \mathbb{R}^n$, conv$\mathbb{R}^n$ is the set of convex compact subsets of $\mathbb{R}^n$; $\rho(l|\mathbf{X}) = \sup\{(x, l)\colon x \in \mathbf{X}\}$, $l \in \mathbb{R}^n$, is the support function of $\mathbf{X} \subset \mathbb{R}^n$; $M^{n\times m}$ is the space of real $n \times m$ matrices; δ_i^j is the Kronecker delta; $E = \{\delta_i^j\}$ is the identity matrix, diagπ, diag $\{\pi_i\}$ is the diagonal matrix whose diagonal elements are the components π_i of a vector π; AbA is the matrix obtained from A by replacing all elements except those on the diagonal by their absolute values; $\top$ is the transposition symbol; $e^i = (0, \dots, 0, 1, 0, \dots, 0)^\top \in \mathbb{R}^n$ is the i-th unit vector in $\mathbb{R}^n$ (with 1 in the i-th position); $e = (1,1,\dots,1)^\top \in \mathbb{R}^n$; $\mathbb{E}(\mathbf{X})$ is the set of all extreme points of the set $\mathbf{X} \in \operatorname{conv}\mathbb{R}^n$ (a point $x \in \mathbf{X}$ is said to be extreme if there are no points $x^1, x^2 \in \mathbf{X}$, $x^1 \neq x^2$, such that $x = (x^1 + x^2)/2$); $\mathbf{B}(c, r)$ is the sphere in $\mathbb{R}^n$ with centre c and radius r; $h_+(\mathbf{X}, \mathbf{Y}) = \min\{\gamma \geqslant 0 | \mathbf{X} \subseteq \mathbf{Y}$

$+ \gamma\mathbf{B}(0,1)\}$ is the Hausdorff half-distance; $y = \max_{x \in \mathbf{X}} \phi(x)$, where $y, \phi \in \mathbb{R}^n$, denotes $y_i = \max_{x \in \mathbf{X}} \phi_i(x)$ $(i = 1, \ldots, n)$.

2. PROPERTIES OF PARALLELEPIPEDS

We will now list a few properties of parallelepipeds (see also [16]).

The support function of a parallelepiped is computed by the formulae

$$\rho(l \mid \mathbf{P}(p, P, \pi)) = (p, l) + \sum_{i=1}^{n} |(p^i, l)| \pi_i$$

An outer estimate for $\mathbf{Q} \in \operatorname{conv} \mathbb{R}^n$, minimal with respect to inclusion among all parallelepipeds with given orientation matrix V, has the form

$$\begin{aligned} &\mathbf{P}_V(\mathbf{Q}) = \mathbf{P}(\upsilon, V, \mathrm{v}) \quad (\upsilon = Vc) \\ &c_i = (\rho((V^{-1})^{\mathrm{T}} e^i \mid \mathbf{Q}) - \rho(-(V^{-1})^{\mathrm{T}} e^i \mid \mathbf{Q}))/2 \\ &\mathrm{v}_i = (\rho((V^{-1})^{\mathrm{T}} e^i \mid \mathbf{Q}) + \rho(-(V^{-1})^{\mathrm{T}} e^i \mid \mathbf{Q}))/2, \quad i = 1, \ldots, n \end{aligned} \tag{2.1}$$

The construction of parallelepipedal estimates for attainability sets is based on performing operations over parallelepipeds (affine transformations, geometrical sum and multiplication by an interval matrix). The result of such an operation may not be a parallelepiped, and in that case it will be approximated from outside by a parallelepiped.

If the matrix $A \in M^{n \times n}$ is non-singular and $a \in \mathbb{R}^n$, then

$$\begin{aligned} &A\mathbf{P}(p, P, \pi) + a = \mathbf{P}(Ap + a, AP, \pi) = \mathbf{P}(Ap + a, APB^{-1}, B\pi) \\ &B = \operatorname{diag}\{\| Ap^i \|\} \end{aligned}$$

For a sum of parallelepipeds, formula (2.1) becomes

$$\mathbf{P}_V\left(\sum_{k=1}^{N} \mathbf{P}(p^{(k)}, P^{(k)}, \pi^{(k)})\right) = \mathbf{P}\left(\sum_{k=1}^{N} p^{(k)}, V, \sum_{k=1}^{N} \operatorname{Abs}(V^{-1}P^{(k)})\pi^{(k)}\right)$$

By an *interval matrix* $\mathbf{A} = \{\mathbf{a}_i^j\}$ given by a pair of matrices $\underline{A} = \{\underline{a}_i^j\}, \bar{A} = \{\bar{a}_i^j\} \in M^{n \times n}, \underline{A} \leqslant \bar{A}$, we mean [14] the matrix whose elements are the intervals $\mathbf{a}_i^j = [\underline{a}_i^j, \bar{a}_i^j]$, or, by another definition, the set of matrices $\mathbf{A} = \{A \in M^{n \times n}: \underline{A} \leqslant A \leqslant \bar{A}\}$. The *product of a set* $\mathbf{X} \subset \mathbb{R}^n$ *by an interval matrix* $\mathbf{A}$ is defined by

$$\mathbf{A} \circ \mathbf{X} = \{y \in \mathbb{R}^n : y = Ax, \ A \in \mathbf{A}, \ x \in \mathbf{X}\}$$

Note that if

$$\tilde{A} = (\underline{A} + \bar{A})/2, \quad \hat{A} = (\bar{A} - \underline{A})/2, \quad \hat{\mathbf{A}} = \{A : \operatorname{Abs} A \leqslant \hat{A}\}$$

then $\mathbf{A} \circ \mathbf{X} \subseteq \tilde{A}\mathbf{X} + \hat{A} \circ \mathbf{X}$. It is well-known [6–8] that even if $\mathbf{X} \in \operatorname{conv} \mathbb{R}^n$, the set $\mathbf{A} \circ \mathbf{X}$ need not be convex.

Let $\mathbf{Q} = \operatorname{co}(\mathbf{A} \circ \mathbf{X})$. If the function $\rho(l \mid \mathbf{Q})$ is known, then formulae (2.1) define a whole family of estimates $\mathbf{P}_V(\mathbf{Q})$. Using previously known results [6], one can verify the truth of the following proposition.

Lemma 1. If $\mathbf{X} \in \operatorname{conv} \mathbb{R}^n$ and $\mathbf{A}$ is an interval matrix, then

$$\rho(l \mid \operatorname{co}(\mathbf{A} \circ \mathbf{X})) = \max_{x \in \mathbf{E}(\mathbf{X})} \sum_{i=1}^{n} \sum_{j=1}^{n} \max\{\underline{a}_i^j l_i x_j, \bar{a}_i^j l_i x_j\} = \max_{x \in \mathbf{E}(\mathbf{X})} \{l^{\mathrm{T}} \tilde{A} x + (\operatorname{Abs} l)^{\mathrm{T}} \hat{A} (\operatorname{Abs} x)\} \tag{2.2}$$

Remark 1. If $\mathbf{X} = \mathbf{P} = \mathbf{P}(p, P, \pi)$, then the maximum in (2.2) is evaluated over all vertices $\mathbf{P}$, of which there are at most 2^n. If the number m of non-zero elements is not large $(m < n)$, it may be less laborious to use another expression for $\rho(l\,|\,\mathrm{co}(\mathbf{A}\circ\mathbf{P}))$. Let $\mathbb{E}(\mathbf{A})$ be the set of "extreme values" of $\mathbf{A}$, that is, the set of all possible different matrices $A^{(k)}$ with elements $a_i^{(k)j} \in \{\underline{a}_i^j, \overline{a}_i^j\}$ $(i,j = 1, \ldots, n)$.

Then

$$\rho(l\,|\,\mathrm{co}(\mathbf{A}\circ\mathbf{P})) = \max_{A^{(k)}\in\mathbb{E}(\mathbf{A})}\left\{(A^{(k)}p,l)+\sum_{i=1}^{n}|(A^{(k)}p^i,l)|\pi_i\right\} \tag{2.3}$$

since

$$\mathrm{co}(\mathbf{A}\circ\mathbf{P}) = \mathrm{co}(\cup\{A^{(k)}\mathbb{E}(\mathbf{P})\,|\,A^{(k)}\in\mathbb{E}(\mathbf{A})\})$$

(see, e.g. [10]). The number of elements $A^{(k)}$ of the set $\mathbb{E}(\mathbf{A})$ is 2^m, where $m \leq n^2$.

We note the following relation between the estimates $\mathbf{P}_E(\mathbf{A}\circ\mathbf{P})$ and the result of using operations from interval analysis [14, 15]. Classical interval arithmetic is an algebraic system $<I(\mathbb{R}), +,-, *,/>$. Its base set $I(\mathbb{R})$ consists of the intervals $[\underline{x}, \overline{x}] = \{x\colon \underline{x} \leq x \leq \overline{x}\}$ of the real axis $\mathbb{R}$. Let $\star \in \{+, -, *, /\}$ be a binary operation on $\mathbb{R}$. If $\mathbf{a}, \mathbf{b} \in I(\mathbb{R})$, then $\mathbf{a} \star \mathbf{b} = \{z = a \star b\colon a \in \mathbf{a}, b \in \mathbf{b}\}$ defines a binary operation on $I(\mathbb{R})$. Let $M^{m\times n}(I(\mathbb{R}))$ be the set of $m \times n$ interval matrices. Operations on this set are defined as follows [14]: if $\mathbf{A} = (\mathbf{a}_i^j)$, $\mathbf{B} = \{b_i^j\} \in M^{m\times n}(I(\mathbb{R}))$, then $\mathbf{A}\pm\mathbf{B} = \{\mathbf{a}_i^j \pm \mathbf{b}_i^j\}$; if $\mathbf{A} \in M^{m\times r}(I(\mathbb{R}))$ and $\mathbf{B} \in M^{r\times n}(I(\mathbb{R}))$, then $\mathbf{A}*\mathbf{B} = \left\{\sum_{k=1}^{r}\mathbf{a}_i^k * \mathbf{b}_k^j\right\}$.

Remark 2. $\mathbf{A}\circ\mathbf{P} \subseteq \mathbf{P}_E(\mathbf{A}\circ\mathbf{P}) \subseteq \mathbf{A}*\mathbf{P}_E(\mathbf{P})$ for any parallelepiped $\mathbf{P}$. But if $\mathbf{P} = \mathbf{P}(p, E, \pi)$, then $\mathbf{P}_E(\mathbf{A}\circ\mathbf{P}) = \mathbf{A}*\mathbf{P}_E(\mathbf{P}) = \mathbf{A}*\mathbf{P}$.

Example. Let us construct a few outer estimates for $\mathbf{A}\circ\mathbf{P}$, where

$$\mathbf{A} = \{A : \mathrm{Abs}(A-\tilde{A}) \leq \hat{A}\},\ \ \tilde{A} = E+\sigma C,\ \ \hat{A} = \sigma D,\ \ \mathbf{P} = \mathbf{P}((1,1)^{\mathrm{T}},\ P,(3,2)^{\mathrm{T}})$$

$$C = \begin{Vmatrix} 0 & 1 \\ -8 & 0 \end{Vmatrix},\ D = \begin{Vmatrix} d & d \\ d & d \end{Vmatrix},\ \ d = 0.5,\ \ \sigma = 0.1,\ \ P = \frac{1}{\sqrt{5}}\begin{Vmatrix} 2 & 1 \\ 1 & 2 \end{Vmatrix}$$

The continuous lines with distinguished vertices in Fig. 1 represent a parallelepiped $\mathbf{P}$, the dashed lines the set $\mathbf{Q} = \mathrm{co}(\mathbf{A}\circ\mathbf{P})$, the x's the points defining $\mathbf{Q}$. The dash–dot lines represent the estimate $\mathbf{A}*\mathbf{P}_E(\mathbf{P})$, the thick lines

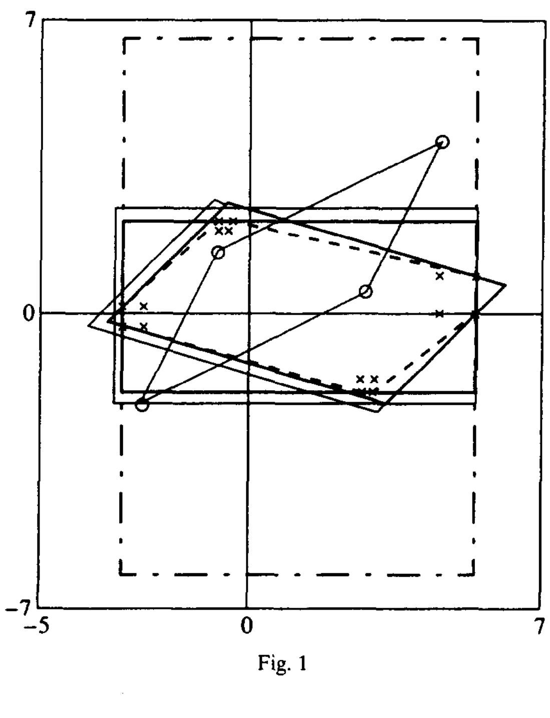

Fig. 1

the estimates $\mathbf{P}_{p(k)}(\mathbf{Q})$ ($k = 1, 2$), where $P^{(1)} = E$ and $P^{(2)}$ is the orientation matrix of the parallelepiped $\tilde{A}\mathbf{P}$. The thin lines represent $\mathbf{P}_{p(k)}(\tilde{A}\mathbf{P} + \hat{\mathbf{A}} \circ \mathbf{P})$ ($k = 1, 2$). It is obvious that estimates of the form $\mathbf{P}_V(\mathbf{A} \circ \mathbf{P})$ and even $\mathbf{P}_V(\tilde{A}\mathbf{P} + \hat{\mathbf{A}} \circ \mathbf{P})$ may prove to be better (in the sense of volume) than $\mathbf{A} * \mathbf{P}_E(\mathbf{P})$.

3. OUTER ESTIMATES OF ATTAINABLE SETS

Let $P(t) \in M^{n \times n}$, $t \in T$ be a continuously differentiable function with

$$\det P(t) \neq 0, \quad t \in T \tag{3.1}$$

($P(t)$ will define the dynamics of orientation matrices). We shall construct outer estimates $\mathbf{P}(t)$ for the attainable set $\mathbf{X}(t)$. The arguments are analogous to those of [3, 4, 7].

Fix $t \in (0, \theta]$ and consider a partition $\mathbf{T}_N$ of the interval $[0, t]$ by points

$$t_0 = 0, \quad t_k = \sum_{i=1}^{k} \sigma_i, \quad k = 1, \dots, N, \quad t_N = t, \ \sigma_i > 0$$

We shall use the simplest finite-difference approximation of system (1.1)

$$\begin{gathered} x[k] = A[k]x[k-1] + B[k]w[k], \quad k = 1, \dots, N \\ A[k] = E + \sigma_k A(t_{k-1}) \in \mathbf{A}[k] = \{E + \sigma_k A: \ A \in \mathbf{A}(t_{k-1})\} \\ x[0] \in \mathbf{X}_0, w[k] \in \mathbf{R}[k] = \sigma_k \mathbf{R}(t_{k-1}), \ B[k] \in \mathbf{B}[k] = \mathbf{B}(t_{k-1}) \end{gathered} \tag{3.2}$$

With the matrices $P[k]$ given, if we construct $\mathbf{P}[k] = \mathbf{P}(p[k], P[k], \pi[k])$,

$$\mathbf{P}[0] = \mathbf{P}_{P[0]}(\mathbf{X}_0), \quad \mathbf{P}[k] = \mathbf{P}_{P[k]}(\mathbf{Z}[k]), \quad k = 1, \dots, N \tag{3.3}$$

$$Z[k] = \sum_{j=1}^{J} \mathbf{Z}^{(j)}[k], \quad \mathbf{Z}^{(1)}[k] = \mathbf{B}[k] \circ \mathbf{R}[k] \tag{3.4}$$

$$\begin{gathered} \mathbf{Z}^{(2)}[k] = \tilde{A}[k]\mathbf{P}[k-1], \quad \mathbf{Z}^{(3)}[k] = \hat{A}[k] \circ \mathbf{P}[k-1], \quad J = 3 \\ \tilde{A}[k] = E + \sigma_k \tilde{A}(t_{k-1}), \quad \hat{\mathbf{A}}[k] = \{\sigma_k A: \ A \in \hat{\mathbf{A}}(t_{k-1})\} \end{gathered} \tag{3.5}$$

then, by what was stated in Section 2, it will be true that

$$X_0 \subseteq \mathbf{P}[0], \quad \mathbf{Z}[k] \subseteq \mathbf{P}[k], \quad k = 1, \dots, N \tag{3.6}$$

and $\mathbf{P}[k]$ will be outer estimates (of type I, superscript I) for the attainability set $\mathbf{X}[k]$ of the finite-difference system (3.2). Similar conditions will be satisfied in general by sharper estimates (of type II, superscript II) of the form (3.3), (3.4), where

$$\mathbf{Z}^{(2)}[k] = \mathbf{A}[k] \circ \mathbf{P}[k-1], \quad J = 2 \tag{3.7}$$

Relying on formulae (3.3)–(3.5) and (3.3), (3.4), (3.7), we construct estimates for $\mathbf{X}(t)$, say $\mathbf{P}^{\mathrm{I}}(t) = \mathbf{P}(p^{\mathrm{I}}(t), P(t), \pi^{\mathrm{I}}(t))$ and $\mathbf{P}^{\mathrm{II}}(t) = \mathbf{P}(p^{\mathrm{II}}(t), P(t), \pi^{\mathrm{II}}(t))$.

As $P[k]$ we take the matrices $P[k] = P(t_k)$. Then

$$P[k] = P[k-1] + \sigma_k \dot{P}(t_{k-1}) + o(\sigma), \quad k = 1, \dots, N \tag{3.8}$$

$$P[k]^{-1} = P[k-1]^{-1} - \sigma_k P[k-1]^{-1} \dot{P}(t_{k-1}) P[k-1]^{-1} + o(\sigma) \tag{3.9}$$

$$\sigma = \max\{\sigma_i \mid i = 1, \dots, N\}, \quad \sigma^{-1} o(\sigma) \to 0 \ \text{ when } \sigma \to 0$$

Applying formulae (3.3)–(3.5) in this specific case, we have

$$p[k] = \sum_{j=1}^{J} p^{(j)}[k], \quad \pi[k] = \sum_{j=1}^{J} \pi^{(j)}[k] \tag{3.10}$$

$$p^{(j)}[k] = P[k](\rho^{(j)+}[k] - \rho^{(j)-}[k])/2, \quad \pi^{(j)}[k] = (\rho^{(j)+}[k] + \rho^{(j)-}[k])/2$$

$$\rho^{(1)\pm}[k] = \sigma_k \max_{w \in \mathbf{E}(\mathbf{R}(t_{k-1}))} \{\pm P[k]^{-1} \tilde{B}(t_{k-1}) w + \mathrm{Abs}(P[k]^{-1}) \hat{B}(t_{k-1}) \,\mathrm{Abs}\, w\}$$

where

$$p^{(2)}[k] = (E + \sigma_k \tilde{A}(t_{k-1})) p[k-1], \quad p^{(3)}[k] = 0$$

$$\pi^{(2)}[k] = \mathrm{Abs}(P[k]^{-1}(E + \sigma_k \tilde{A}(t_{k-1})) P[k-1]) \pi[k-1] \tag{3.11}$$

$$\pi^{(3)}[k] = \sigma_k \max_{x \in \mathbf{E}(\mathbf{P}[k-1])} \mathrm{Abs}(P[k]^{-1}) \hat{A}(t_{k-1}) \,\mathrm{Abs}\, x$$

We now subtract $\pi[k-1]$ from both sides of the expression for $\pi[k]$, divide by σ_k and take the limit as $\sigma \to 0$. Throughout we use expression (3.9), estimates of the type

$$\| a + \varepsilon | - | a \| \leqslant 3 | \varepsilon |, \quad | \max\{a_1 + \varepsilon_1, a_2 + \varepsilon_2\} - \max\{a_1, a_2\} | \leqslant \max\{| \varepsilon_1 |, | \varepsilon_2 |\}$$

the equality

$$\mathrm{Abs}(E + D) - E = \mathrm{Ab}\, D, \quad \forall D = \{d_i^j\} : | d_i^i | \leqslant 1, \quad i = 1, \dots, n$$

and the continuity of $\tilde{A}$, $\hat{A}$, $\underline{B}$, $\bar{B}$. The result is a system of ordinary differential equations (ode)

$$\dot{\pi}^1 = \mathrm{Ab}(P^{-1}(\tilde{A}P - \dot{P}))\pi^1 + \max_{\xi \in \mathbf{E}(\mathbf{P}(0,E,e))} \mathrm{Abs}(P^{-1})\hat{A} \ \mathrm{Abs}(p^1 + P \,\mathrm{diag}\, \pi^1 \xi) + (f^+ + f^-)/2 \tag{3.12}$$

$$f^{\pm} = \max_{w \in \mathbf{E}(\mathbf{R}(t))} \{\pm P^{-1} \tilde{B} w + \mathrm{Abs}(P^{-1}) \hat{B} \,\mathrm{Abs}\, w\} \tag{3.13}$$

where for brevity the arguments t are omitted in all terms. Similarly

$$\dot{p}^1 = \tilde{A} p^1 + P(f^+ - f^-)/2, \quad t \in T \tag{3.14}$$

Now applying formulae (3.3), (3.4) and (3.7) in this specific case, we obtain (3.10), where

$$\rho_i^{(2)\pm}[k] = \max_{\xi \in \mathbf{E}(\mathbf{P}(0,E,e))} \sum_{\alpha,\beta=1}^{n} \max_{l=1,2} \{\pm \psi_{i\alpha\beta}^{(l)}(\sigma_k, \xi)\} \tag{3.15}$$

Here, taking (3.9) into consideration, we have

$$\psi_{i\alpha\beta}^{(1)}(\sigma, \xi) = (P^{-1} - \sigma P^{-1} \dot{P} P^{-1} + o(\sigma))_i^{\alpha} (E + \sigma \underline{A})_{\alpha}^{\beta} (p + P \,\mathrm{diag}\, \pi \xi)_{\beta}$$

where the arguments t_{k-1} and $k-1$ are omitted for brevity. Separating terms that are small to the first order in σ, we introduce the notation

$$\varphi_{i\alpha\beta}^{(1)}(\xi) = (-(P^{-1} \dot{P} P^{-1})_i^{\alpha} \delta_{\alpha}^{\beta} + (P^{-1})_i^{\alpha} \underline{a}_{\alpha}^{\beta})(p + P \,\mathrm{diag}\, \pi \xi)_{\beta} \tag{3.16}$$

The functions $\psi_{i\alpha\beta}^{(2)}(\sigma, \xi)$ and $\varphi_{i\alpha\beta}^{(2)}(\xi)$ are obtained from $\psi_{i\alpha\beta}^{(1)}(\sigma, \xi)$ and $\varphi_{i\alpha\beta}^{(1)}(\xi)$ by replacing $\underline{A}$ by $\bar{A}$. But $\pi^{(2)}[k-1] = (\tilde{\rho}^{(2)+} + \tilde{\rho}^{(2)-})/2$, where

$$\tilde{\rho}_i^{(2)\pm} = (\pm P^{-1} p + \pi)_i = \max_{\xi \in \mathbf{E}(\mathbf{P}(0,E,e))} \pm e^{i^{T}} P^{-1}(p + P \,\mathrm{diag}\, \pi \xi) \tag{3.17}$$

that is, $\tilde{\rho}_i^{(2)\pm}$ are identical with the right-hand sides of Eqs (3.15) for $\sigma_k = 0$. Using arguments analogous to those of [17, pp. 71–72], it can be shown that

$$\rho_i^{(2)\pm}[k] - \tilde{\rho}_i^{(2)\pm} = \sigma_k \Phi_i^{\pm} + o(\sigma), \quad i = 1,\dots,n \tag{3.18}$$

$$\Phi_i^{\pm} = \max_{\xi \in \Xi_i^{\pm}} \sum_{\alpha,\beta=1}^{n} \max_{l=1,2}\{\pm\varphi_{i\alpha\beta}^{(l)}(\xi)\} = \max_{\xi \in \Xi_i^{\pm}}(\pm P^{-1}(\tilde{A} - \dot{P}P^{-1})x +$$

$$+ \operatorname{Abs}(P^{-1})\hat{A} \operatorname{Abs} x)_i; \quad x = p + P \operatorname{diag} \pi \xi$$

In view of condition (3.1) and the continuity of $\underline{A}$ and $\bar{A}$, we have here $\sigma^{-1}o(\sigma) \to 0$ as $\sigma \to 0$, uniformly in t, p, π, if the latter are chosen in a closed bounded domain. The maximum in (3.18) must be evaluated over those $\xi \in \mathbb{E}(\mathbf{P}(0, E, e))$ at which the maximum in (3.17) is achieved, that is, one can take

$$\Xi_i^{\pm} = \{\xi : \xi \in \mathbb{E}(\mathbf{P}(0, E, e)), \quad \xi_i = \pm 1\} \tag{3.19}$$

Writing out $(\pi[k] - \pi[k-1])/\sigma_k$ and $(p[k] - p[k-1])/\sigma_k$ using formulae (3.10), (3.8), (3.18) and (3.17) and letting $\sigma \to 0$, we arrive at a system of ode

$$\begin{aligned} \dot{\pi}^{\mathrm{II}} &= (\Phi^+ + \Phi^-)/2 + (f^+ + f^-)/2 \\ \dot{p}^{\mathrm{II}} &= \dot{P}P^{-1}p^{\mathrm{II}} + P(\Phi^+ - \Phi^-)/2 + P(f^+ - f^-)/2 \end{aligned} \tag{3.20}$$

Theorem 1. Let $\mathbf{X}(t)$ be the attainability set of system (1.1)–(1.5), (1.8), and let $P(t) \in M^{n\times n}$, $t \in T$ be an arbitrary given continuously differentiable function satisfying condition (3.1). If the parameters of the parallelepipeds $\mathbf{P}^{\mathrm{I}}(t) = \mathbf{P}(p^{\mathrm{I}}(t), P(t), \pi^{\mathrm{I}}(t))$ are defined by the ode's (3.12)–(3.14) (of type I) and

$$\pi^{\mathrm{I}}(0) = \operatorname{Abs}(P(0)^{-1}P_0)\pi_0, \quad p^{\mathrm{I}}(0) = p_0 \tag{3.21}$$

then $\mathbf{P}^{\mathrm{I}}(t)$ will satisfy relations (1.9) and (1.10). The same holds for $\mathbf{P}^{\mathrm{II}}(t)$ as defined by Eqs (3.21) (with the superscript I replaced by II) and ode's (3.20) (of type II), where the dependence of the functions $f^{\pm}$ on t and $\Phi^{\pm}$ on $t, p^{\mathrm{II}}, \pi^{\mathrm{II}}$ is defined by formulae (3.13), (3.18), (3.19) and (3.16) with p and π replaced by p^{II} and π^{II}. In addition, $\mathbf{X}(t) \subseteq \mathbf{P}^{\mathrm{II}}(t) \subseteq \mathbf{P}^{\mathrm{I}}(t)$.

The proof is analogous to that of [4, Section 8.3] (but without the change of variables of [4, p. 129]). To verify inclusions (1.9), one uses the inclusion

$$\mathbf{X}(t_k, t_j, \mathbf{P}[j]) \subseteq \mathbf{P}[k] + o(1)\mathbf{B}(0,1), \quad t_k, t_j \in \mathbf{T}_N$$

where $\mathbf{P}[k]$ are constructed according to formulae (3.3)–(3.5) or (3.3), (3.4) and (3.7), $o(1) \to 0$ as $\sigma \to 0$, the quantity $o(1)$ is uniformly small on T and is independent of $\mathbf{T}_N$. For the proof of the last inclusion, one uses the integral funnel equation [18, 8] to establish that

$$h_+(\mathbf{X}(t_{i+1}, t_i, \mathbf{P}[i]), \ \mathbf{P}[i+1]) \leqslant o(\sigma)$$

and then reasons along the same lines as in [19, Chapter VIII, Section 4]. The relations $\mathbf{P}^{\mathrm{II}}[k] \subseteq \mathbf{P}^{\mathrm{I}}[k]$ guarantee the truth of the inclusions $\mathbf{P}^{\mathrm{II}}[t] \subseteq \mathbf{P}^{\mathrm{I}}[t]$.

Remark 3. Outer ellipsoidal estimates have been constructed [7] for the attainability set of system (1.1), (1.2), (1.5), where $B(t) \equiv E$, $\mathbf{X}_0$ is an ellipsoid and the function $w(t)$ is known exactly. The ode's for the matrix of the ellipsoid require a maximum operation requiring the checking of 2^{m-1} versions, where m is the number of non-zero elements of $\hat{A}$. In the general case, $m = n^2$, and then the computation of parallelepipedal estimates may prove to be less laborious.

Remark 4. If the number of non-zero elements of $\hat{A}$ and $\hat{B}$ is not large, the following changes may be made in Eqs (3.12) and (3.13)

$$\max_{x \in \mathbf{E}(\mathbf{P}^{\mathrm{I}}(t))} \operatorname{Abs}(P^{-1})\hat{A} \operatorname{Abs} x = \max_{D \in \mathbf{E}(\hat{\mathbf{A}}(t))} \{P^{-1}Dp^{\mathrm{I}} + \operatorname{Abs}(P^{-1}DP)\pi^{\mathrm{I}}\}$$

$$f^{\pm} = \max_{B \in \mathbf{E}(\mathbf{B}(t))} \{\pm P^{-1}Br + \operatorname{Abs}(P^{-1}BR)\rho\}$$

Remark 5. If $\underline{B} \equiv \bar{B} \equiv E$, then

$$(f^+ + f^-)/2 = \mathrm{Abs}(P^{-1}R)\rho, \quad P(f^+ - f^-)/2 = r$$

If moreover $\underline{A} \equiv \bar{A}$, then system (3.12), (3.14) is identical with system (3.20).

Remark 6. If $P(t)$ is evaluated by a system

$$\dot{P} = \tilde{A}P, \quad t \in T; \quad P(0) = P_0 \tag{3.22}$$

then, under the assumptions of Remark 5, one arrives at the equations derived in [16].

Remark 7. If $P \equiv E$, we obtain coordinatewise estimates of two types for $\mathbf{X}(t)$. If

$$\mathbf{X}_0 = \mathbf{P}(0, E, \pi_0), \quad \mathbf{R} \equiv \mathbf{P}(0, E, \rho), \quad \tilde{A} \equiv \mathrm{diag}\lambda, \quad \underline{B} \equiv \bar{B} \equiv E, \quad P \equiv E$$

Eqs (3.12) and (3.14) are identical with those derived in [20], by another method, under the same assumptions.

Remark 8. If $\tilde{A}$ is a constant simple matrix with n real eigenvalues λ_i and the columns of P are constant and identical with the eigenvectors of $\tilde{A}$, that is,

$$P \mathrm{diag}\lambda = \tilde{A}P$$

then

$$\dot{\pi}^{\mathrm{I}} = \mathrm{diag}\lambda\pi^{\mathrm{I}} + \max_{\xi \in \mathbf{E}(\mathbf{P}(0,E,e))} \mathrm{Abs}(P^{-1})\hat{A}\,\mathrm{Abs}(p^{\mathrm{I}} + P\mathrm{diag}\pi^{\mathrm{I}}\xi) + \frac{f^+ + f^-}{2}$$

If the elements of $\hat{A}$ are not "small", then even sharper estimates $\mathbf{P}^{\mathrm{II}}(t)$ may turn out to be too coarse. We shall demonstrate a case in which the attainability set may be estimated more accurately by combining such estimates.

Corollary 1. Suppose the system belongs to the class of systems of constant coefficients of type (1.1)–(1.4), where

$$A(t) \equiv A, \quad \mathbf{A}(t) \equiv \mathbf{A}, \quad \hat{B}(t) \equiv 0 \tag{3.23}$$

and $\mathbf{Y}(t)$ is the attainability set of system (1.1)–(1.4), (3.23). (For $\mathbf{A}(t) \equiv \mathbf{A}$, attainability sets $\mathbf{X}(t)$ and $\mathbf{Y}(t)$ corresponding to two different cases, when the matrix A is assumed to be dependent on or independent of t, are generally different.) Let

$$\mathbf{A} \subseteq \bigcup_{j=1}^{J} \mathbf{A}^{(j)} \tag{3.24}$$

that is, for each matrix $A \in \mathbf{A}$ an interval matrix $\mathbf{A}^{(j)}$ exists such that $A \in \mathbf{A}^{(j)}$. Let $\mathbf{X}^{(j)}(t)$ and $\mathbf{Y}^{(j)}(t)$ denote the attainability sets of systems (1.1)–(1.3) and (1.1)–(1.4), (3.23), respectively, where the matrix $\mathbf{A}$ is replaced by $\mathbf{A}^{(j)}$, and let $\mathbf{P}^{(j)}(t)$ denote estimates of type I or II for $\mathbf{X}^{(j)}(t)$ corresponding to $P^{(j)}(\cdot)$. Then

$$\mathbf{Y}(t) \subseteq \bigcup_{j=1}^{J} \mathbf{Y}^{(j)}(t) \subseteq \bigcup_{j=1}^{J} \mathbf{X}^{(j)}(t) \subseteq \bigcup_{j=1}^{J} \mathbf{P}^{(j)}(t), \quad t \in T \tag{3.25}$$

Remark 9. Inclusion (3.24) may be obtained, for example, by introducing an "interval lattice". Namely, expressing the elements $\mathbf{a}_\alpha^\beta$ of the matrix $\mathbf{A}$ in the form

$$\mathbf{a}_\alpha^\beta = \bigcup_{j_{\alpha\beta}=1}^{J_{\alpha\beta}} \mathbf{a}^{(j_{\alpha\beta})\beta}{}_\alpha$$

we introduce the set of all possible interval matrices with elements $\mathbf{a}^{(j_{\alpha\beta})\beta}{}_\alpha$. The number of such matrices is

$$J = \prod_{\alpha,\beta=1}^{n} J_{\alpha\beta}$$

where $J_{\alpha\beta} = 1$ for one-point intervals $\mathbf{a}_\alpha^\beta$. For a "sufficiently dense" "interval lattice", outer estimates for $\mathbf{Y}(t)$ may be made more accurate by taking the intersection of estimates (3.25) over certain sets $\Pi^{(j)}$ of the functions $P^{(j)}(\cdot)$

$$\mathbf{Y}(t) \subseteq \bigcup_{j=1}^{J} \bigcap_{P^{(j)}(\cdot) \in \Pi^{(j)}} \mathbf{P}^{(j)}(t)$$

In particular, finding $P^{(j)}(\cdot)$ by using system (3.22), one can take the intersection over some set of matrices $P(0) = P^0$

$$\mathbf{Y}(t) \subseteq \bigcap_{P^0} \bigcup_{j=1}^{J} \mathbf{P}^{(j)}(t)$$

4. EXAMPLES

1. Let

$$\tilde{A}(t) \equiv \begin{Vmatrix} 0 & 1 \\ -1 & 0 \end{Vmatrix}, \quad \hat{A}(t) \equiv \begin{Vmatrix} 0 & 0 \\ 0.8 & 0 \end{Vmatrix}, \quad \underline{B}(t) \equiv \overline{B}(t) \equiv \begin{Vmatrix} 0 & 0 \\ 0 & 0 \end{Vmatrix}, \quad \theta = 2$$

Let $\mathbf{P}^{(1)}(t)$, $\mathbf{P}^{(2)}(t)$ and $\mathbf{P}^{(3)}(t)$ denote estimates $\mathbf{P}^{\mathrm{II}}(t)$ for attainability sets corresponding to $\mathbf{X}_0 = \mathbf{P}_0^{(1)} = \mathbf{P}(0, E, e)$ (the first case), $\mathbf{X}_0 = \mathbf{P}_0^{(2)} = \mathbf{P}(0, E, (0.1, 1)^{\mathrm{T}})$ (the second case) and $\mathbf{X}_0 = \mathbf{P}_0^{(3)} = \mathbf{P}(0, E, (1, 0.1)^{\mathrm{T}})$ (the third case), obtained assuming that relations (3.22) hold. Let $\mathbf{E}^{(1,1)}(t)$, $\mathbf{E}^{(1,2)}(t)$, $\mathbf{E}^{(2)}(t)$ and $\mathbf{E}^{(3)}(t)$ be ellipsoidal outer estimates for attainability sets corresponding to $\mathbf{X}_0 = \mathbf{E}_0^{(1,1)} = \mathbf{B}(0, 1)$, $\mathbf{X}_0 = \mathbf{E}_0^{(1,2)} = \mathbf{B}(0, \sqrt{2})$, $\mathbf{X}_0 = \mathbf{E}_0^{(2)} = \mathbf{E}(0, \mathrm{diag}\{0.01, 1\})$ and $\mathbf{X}_0 = \mathbf{E}_0^{(3)} = \mathbf{E}(0, \mathrm{diag}\{1, 0.01\})$, constructed using the equations from the example of [7]. Here $\mathbf{E}(q, Q) = \{x : (Q^{-1}(x-q), (x-q)) \leq 1\}$ denotes an ellipsoid.

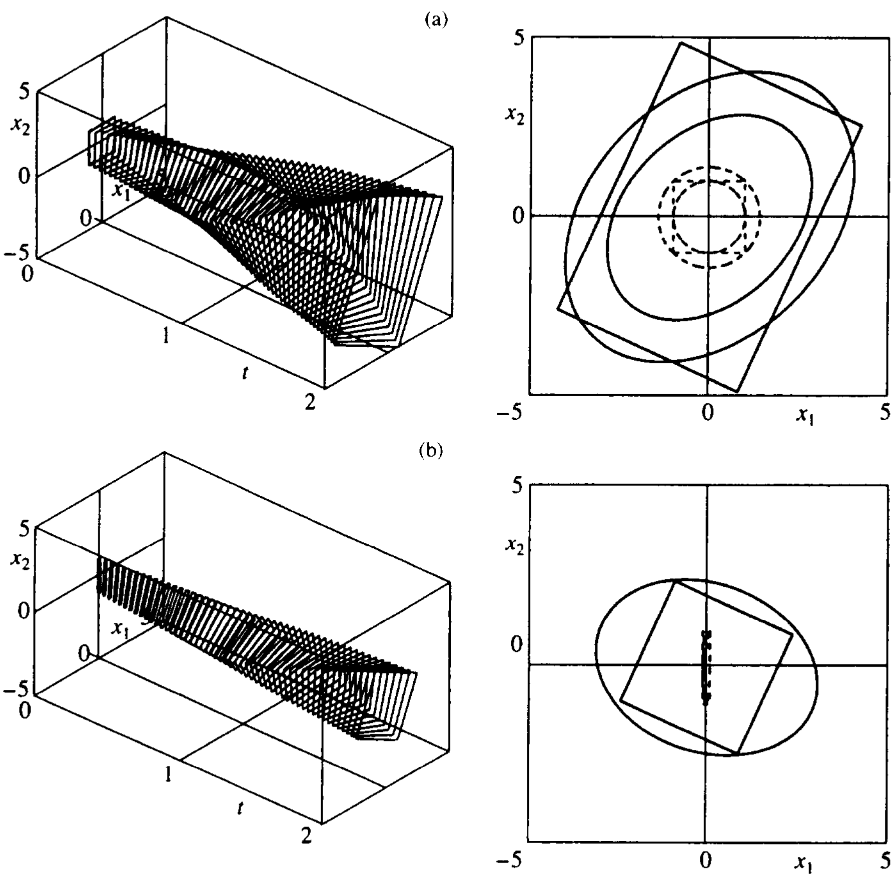

Fig. 2

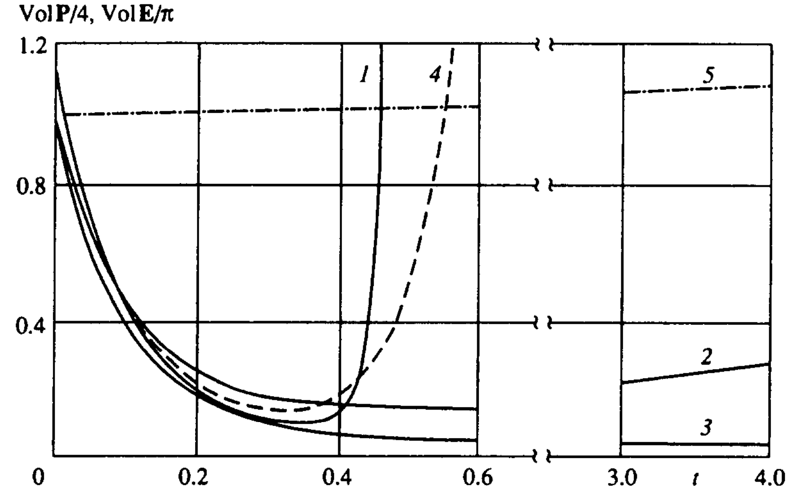

Fig. 3

Numerical calculation of $\mathbf{P}^{(i)}$ is done using formulae (3.3), (3.4) and (3.7) for $N = 500$, and ellipsoidal estimates are found using the numerical integration function ode23.m of the MATLAB system. In Fig. 2(a) we show on the left sections of $\mathbf{P}^{(1)}(\cdot)$ for every 10 steps k; on the right, shown by the dashed lines, are the sets $\mathbf{E}_0^{(1,1)} \subset \mathbf{P}_0^{(1)} \subset \mathbf{E}_0^{(1,2)}$, and by the solid lines – the estimates $\mathbf{E}^{(1,1)}(\theta)$, $\mathbf{P}^{(1)}(\theta)$ and $\mathbf{E}^{(1,2)}(\theta)$ (parallelogram – $\mathbf{P}^{(1)}(\theta)$, inner ellipse – $\mathbf{E}^{(1,1)}(\theta)$, outer ellipse – $\mathbf{E}^{(1,2)}(\theta)$). Figure 2(b) corresponds to the second case. The table lists the ratios of the volumes of the estimates at certain instants of time to the volumes $\mathbf{X}_0$. Comparison of the results as represented by Fig. 2 and the table shows that parallelepipedal estimates have proved comparable with ellipsoidal estimates: for the first case they are inferior, but for the second and third – better (in particular, $\mathbf{P}^{(2)}(\theta) \subset \mathbf{E}^{(2)}(\theta)$, although $\mathbf{P}_0^{(2)} \supset \mathbf{E}_0^{(2)}$). The "wrapping effect" for the estimates described by ode (3.20) and (3.22) has turned out to be significantly less than for coordinatewise estimates, as described by ode (3.2) with $P \equiv E$ (see the table, where $\mathbf{P}^{(i)c}(t)$ denote coordinatewise estimates for the cases $i = 1, 2, 3$). In this example, the parallelepipedal estimates of types I and II were the same. For $\mathbf{X}_0 = \mathbf{P}(e, E, e)$ the volumes of the estimates of types I and II, when relations (3.22) hold, differ by a factor of more than 1.5 at time θ.

2. Let

$$\bar{A}(t) \equiv \begin{Vmatrix} 0 & 1 \\ -1 & -15 \end{Vmatrix}, \quad \hat{A}(t) \equiv \begin{Vmatrix} 0 & 0 \\ 0 & 5 \end{Vmatrix}, \quad \underline{B}(t) \equiv \bar{B}(t) \equiv \begin{Vmatrix} 0 & 0 \\ 0 & 0 \end{Vmatrix}, \quad \theta = 4$$

(a system with constant coefficients in the selected range describes a damped harmonic oscillator).

i	1		2		3	
t	1	2	1	2	1	2
$\operatorname{vol}\mathbf{P}^{(i)}(t)/\operatorname{vol}\mathbf{P}_0^{(i)}$	3.8	11.4	6.3	32.8	13.8	33.6
$\operatorname{vol}\mathbf{E}^{(1,j)}(t)/\operatorname{vol}\mathbf{E}_0^{(1,j)},\ j=1,2$	2.6	7.7				
$\operatorname{vol}\mathbf{E}^{(i)}(t)/\operatorname{vol}\mathbf{E}_0^{(i)}$			7.0	74.2	27.4	53.1
$\operatorname{vol}\mathbf{P}^{(i)c}(t)/\operatorname{vol}\mathbf{P}_0^{(i)}$	14.9	218.1	35.0	511.8	56.5	826.8

We shall construct estimates $\mathbf{P}^{(i)}(t)$ $(i = 1, 2, 3)$ for the attainability set for $\mathbf{X}_0 = \mathbf{P}(0, E, e)$ in accordance with Remarks 6–8 (the estimates of types I and II for the first two cases are again identical) and find estimates $\mathbf{E}^{(4)}(t)$ and $\mathbf{E}^{(5)}(t)$ for the attainability sets for $\mathbf{X}_0 = \mathbf{B}(0, 2)$, using the ode's of [7] and the technique of [10], respectively.

The graphs in Fig. 3 show the ratio of the volume of the estimate to the volume of $\mathbf{X}_0$. The volume of the ellipsoidal estimate [7] (curve *4*) first decreases and then increases, reaching the value 3.4×10^{22} at time θ. The algorithm proposed in [10] yields only a stable ellipsoidal estimate, whose volume increases only slightly (see curve *5*; it was derived in [10] for a computation with step-size $\theta/N = 10^{-4}$). The estimate $\mathbf{P}^{(1)}(\cdot)$ (curve *1*) could be constructed up to time θ neither by using formulae (3.3), (3.4) and (3.7) with $N = 10^4$, nor by integrating the ode's by means of the solver system of MATLAB, since the matrix $P(t)$ approaches a singular matrix. The volume of the

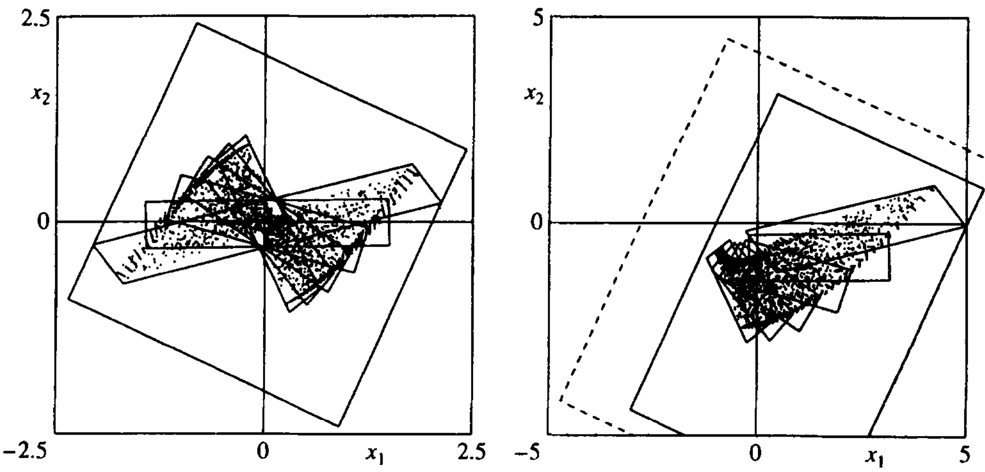

Fig. 4

coordinatewise estimate $\mathbf{P}^{(2)}(\cdot)$ (curve *2*) first decreases and then slowly increases. On the whole, the best results (in the sense of volume) are given by the estimate $\mathbf{P}^{(3)}(\cdot)$ (curve *3*), whose volume decreases monotonically (although $\mathrm{vol}\mathbf{P}^{(3)}(0)/\mathrm{vol}\mathbf{X}_0 > 1$). Thus, the estimates $\mathbf{P}^{(2)}(\cdot)$ and $\mathbf{P}^{(3)}(\cdot)$ have proved to be better than the ellipsoidal estimates proposed in [7, 10].

3. Let us consider Example 1 with the additional assumption that $A(t) \equiv A$. Figure 4 shows points of the attainability set $\mathbf{Y}(\theta)$ corresponding to $\mathbf{X}_0 = \mathbf{P}_0^{(2)}$ (left) and $\mathbf{X}_0 = \mathbf{P}_0^{(2)} + e$ (right). These points were computed using Cauchy's formula for randomly chosen initial points of $\mathbf{X}_0$ and values of the matrix A from $\mathbf{A}$. The "large" parallelepipeds are outer estimates for the sets $\mathbf{Y}(\theta) \subseteq \mathbf{X}(\theta)$, found on the basis of condition (3.22) (estimates of types I and II are represented by the dashed and solid lines, respectively; in the diagram on th left they coincide). The figure also shows parallelepipeds of type II (assuming the truth of condition (3.22)) for systems corresponding to the "interval lattice" obtained by partitioning $\mathbf{a}_2^1 = [-1.8, -0.2]$ into six equal parts. The combination of these parallelepipeds yields more accurate non-convex estimates for $\mathbf{Y}(\theta)$.

I wish to thank A. B. Kurzhanskii for his interest, comments and useful discussions, and also B. I. Anan'yev, M. I. Gusev and T. F. Filippova for their suggestions and comments.

This research was supported financially by the Russian Foundation for Basic Research (00-01-00369).

REFERENCES

1. KURZHANSKII, A. B., *Control and Observation Under Uncertainty Conditions*. Nauka, Moscow, 1977.
2. KRASOVSKII, N. N. and SUBBOTIN, A. I., *Positional Differential Games*. Nauka, Moscow, 1974.
3. KURZHANSKII, A. B. and VALYI, I., *Ellipsoidal Calculus for Estimation and Control*. Birkhäuser, Boston, 1997.
4. CHERNOUS'KO, F. L., *Estimation of the Phase State of Dynamical Systems. The Method of Ellipsoids*. Nauka, Moscow, 1988.
5. KURZHANSKII, A. B., Dynamic control system estimation under uncertainty conditions, I. *Prob. Control and Inform. Theory*, 1980, **9**, 6, 395–406; II, ibid., 1981, **10**, 1, 33–42.
6. BARMISH, B. R. and SANKARAN, J., The propagation of parametric uncertainty via polytopes. *IEEE Trans. Automat. Control*, AC-24, 1979, 2, 346–349.
7. CHERNOUS'KO, F. L., Ellipsoidal approximation of attainability sets of a linear system with an indeterminate matrix. *Prikl. Mat. Mekh.*, 1996, **60**, 6, 940–950.
8. FILIPPOVA, T. F. and LISIN, D. V., On the estimation of trajectory tubes of differential inclusions. In *Proc. Steklov Inst. Math.*, Suppl. 2, 2000, 28–37.
9. MILANESE, M., NORTON, J., PIET-LAHANIER, H. and WALTER, E., (Eds.), *Bounding Approaches to System Identification*. Plenum Press, New York and London, 1996.
10. KAYUMOV, R. I., Guaranteed estimates of the state of a class of systems with uncertain coefficients. In *Estimation of the Dynamics of Controlled Motions*. Ural. Otd. Akad. Nauk SSSR, Sverdlovsk, 1988, 57–64.
11. ROKITYANSKII, D. Ya., Optimal ellipsoidal estimates of the attainability set of linear systems with an uncertain matrix. *Izv. Ross. Akad. Nauk. Teoriya i Sistemy Upravleniya*, 1997, 4, 17–20.
12. KORNOUSHENKO, Ye. K., Interval coordinatewise estimates for the set of attainable states of a linear stationary system, I. *Avtomatika i Telemekhanika*, 1980, 5, 12–22; II, ibid., 1980, 12, 10–17; III, ibid., 1982, 10, 47–52; IV, ibid., 1983, 2, 81–87.
13. KOSTOUSOVA, E. K. and KURZHANSKII, A. B., Guaranteed estimates of accuracy of computations in control and estimation problems. *Vychisl. Tekhnologii*, 1997, **2**, 1, 19–27.
14. ALEFELD, G. and HERZBERGER, J., *Introduction to Interval Computations*. Academic Press, New York, 1983.
15. KALMYKOV, S. A., SHOKIN, Yu. I. and YULDASHEV, Z. Kh., *Methods of Interval Analysis*. Nauka, Novosibirsk, 1986.
16. KOSTOUSOVA, E. K., State estimation for dynamic systems via parallelotopes: Optimization and parallel computations. *Optimization Methods and Software*, 1998, **9**, 4, 269–306.

17. DEM'YANOV, C. F. and MALOZEMOV, V. N., *Introduction to Minimax*, Nauka, Moscow, 1972.
18. PANASYUK, A. I. and PANASYUK, V. I., An equation generated by a differential inclusion. *Mat. Zametki*, 1980, **27**, 3, 429–437.
19. BAKHVALOV, N. S., ZHIDKOV, N. P. and KOBEL'KOV, G. M., *Numerical Methods*. Nauka, Moscow, 1987.
20. ROKITYANSKII, D. Ya., Evolution equations of optimal estimates of the attainability set of linear systems with an uncertain matrix. *Dokl. Ross. Akad. Nauk*, 1999, **364**, 5, 608–610.

Translated by D.L.